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# Forest Management in an Urbanizing Landscape

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# Forest Management in an Urbanizing Landscape

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## Abstract

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This paper aims at building a theoretical framework to examine the impact of development pressure on private owner's forest management practices, namely, on regeneration and conversion cut dates. As the rent for developed land is rising over time, our model creates the possibility of switching from forestry to residential use at some point in the future, thus departing from the Faustmann's traditional setup. Comparative statics results with respect to stumpage prices, regeneration costs and urban growth parameters are provided. The results obtained depend on the impact on the opportunity cost of holding the stand and the impact on the opportunity cost of holding the land, generalizing Faustmann's unambiguous results.

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**Keywords:** urban growth; increasing residential rents; forest management practices

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## 1. Introduction

In the United States, concern about the effects of development on private forests has risen sharply since the 1990's, when the conversion of forestland to urban uses reached a million acres per year. Nationwide, more than 60% of US housing units built during the 1990 were constructed on or near areas of wildland vegetation. Between 1982 and 1997, the US population grew by 17%, while the urbanized area rose by 47% (Alig (2010)). A total of 49.7 million acres of forest across the U.S. is now projected to be converted to urban use by 2062 (Alig et al. (2010)).

Population and personal incomes growth are the primary forces driving changes in the forest landscape (Kline et al. (2004), Alig et al. (2010)). By increasing the demand for land in residential use and therefore its value, these forces create pressure for foresters to either sell or convert their land to urban use. On the other hand, rising prices for wood products tend to increase the relative rents associated with keeping the land in forest rather than converting it to an alternative use. Yet, revealed behavior by landowners in fast urbanizing areas indicates that values for residential uses are generally higher than those for forestry and agriculture (Alig and Plantinga (2004), Alig et al. (2004)). One thus may expect traditional timber management to become a transitional use on the wildland-urban and rural-urban interfaces in urbanizing regions.<sup>1</sup>

This naturally raises the following questions: What are the implications for the Faustmann harvesting strategy when conversion to an irreversible land use may occur at some point in the future? What are the impacts on private harvesting and conversion decisions of an increase in the growth rate of urban rents?

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<sup>1</sup> The wildland-urban interface is a zone of transition between unoccupied land and human development. Lands and communities that are adjacent to or surrounded by wildlands are at risk of wildfires. On the other hand, the rural-urban interface is defined as a transition zone where urban and rural uses mix.

This paper aims to build a theoretical framework for determining the optimal regeneration and conversion cut dates in a rapidly urbanizing region. In particular, we develop a model of a forest owner where the rent for developed land is rising over time. In addition, we present a rigorous analysis of the impact of changes in the stumpage price, regeneration costs and urban growth parameters on optimal harvesting dates and optimal rotation lengths.

The inclusion of rotation-end land sale or of a change to a more profitable postharvest land use has seldom been analyzed in the optimal rotation literature. Still, the fact that urban rents change over time alters the nature of the timber problem since it forces us to allow for the possibility to changes in land use from timber to urban use at some point in the future. Besides, the incorporation of future conversion to higher valued uses in the optimal harvest condition derived from the classic Faustmann formulation clearly affect forest management practices and the decision of land use change.

Within the context of our model, we show that the optimal harvest length may vary over time when a change in future land use occurs due to rising urban rents. Regardless whether optimal rotation lengths are increasing, constant or decreasing over time, we also show that the optimal length of the harvesting cycles is always lower than the Faustmann rotation length. This suggests that timber production and active forest management might decline or change in some areas as a consequence of increased development pressure.

Additionally, our comparative statics analysis reveals that an increase in the current urban rental rate or in the expected rate of growth of urban rental rates or a decrease in conversion costs hastens conversion and shortens the regeneration cut date because it makes forest management relatively less profitable. Yet, changes in stumpage prices or

in the discount rate always lead to the possibility of uncertain results. This contrasts with the comparative statics from the basic Faustmann model which indicate that, in general, if timber prices rise, then harvest rotation lengths shorten. Similarly, if the discount rate rises then rotation lengths decline.

The rationale behind our previous result rests on the fact that when the number of rotation cycles is finite and landowners do not engage in long rotation timber farming, the impact on the opportunity cost of holding the land of a change in stumpage price or in the discount rate may not be negligible. Therefore, both the impact on the opportunity cost of holding the stand and the impact on the opportunity cost of holding the land play a role in determining regeneration and conversion cut dates. Because these effects run in opposite directions and depend on market conditions, one cannot have simple unambiguous results as in the Faustmann case.

The rest of the paper is organized as follows. Our next section provides a background discussion focusing on theoretical models of forest management practices that have either considered the case of land sale following clear cutting or the case where the landowner can switch to a more profitable alternative land use without selling the land. Section 3 develops our analytical model, describes the solution of the private landowner and discusses how the optimal harvesting strategy changes from the Faustmann setup. Section 4 presents the comparative statics results. Finally, section 5 offers conclusions.

## **2. Literature Review**

The traditional Faustmann setup investigates the optimal harvesting strategy for successive timber crops under the assumption that stumpages prices and regeneration

costs remain constant over time and disregards the existence of any alternative use to forestry. Basically, the setup assumes that the value in current best use of land and the current best use will prevail for succeeding uses of land. This implies that land will be perpetually used for timber production and that rotation lengths are constant over time.

Very few theoretical studies have investigated the implications for the Faustmann strategy of forestland conversion over time. To our knowledge there have been only two studies that have examined this issue (McConnell et al. (1983), Burgess and Ulph (2001)). McConnell et al. (1983) determine the “approximately” optimal harvesting strategy of a forester who maximizes the present discounted value net revenue from a single site when timber prices and regeneration costs vary exogenously and agricultural rents remain constant over time.<sup>2</sup> Burgess and Ulph (2001) use a forest land use option model to allow for the conversion of forestland between alternative management options over time to explain the ongoing process of deforestation in the tropics.

More specifically, McConnell et al. (1983) deal with the possibility of shifting from forestry to agriculture and vice-versa, while Burgess and Ulph (2001) focus on the switch between alternative valued tree crops over time. Therefore, none of these previous frameworks is suitable to understand current trends in forestland conversion in areas with strong urban growth pressure. Yet, both studies provide important insights on how

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<sup>2</sup> Armstrong and Philips (1989) also develop a theoretical framework to determine the optimal timing of land use change from timber production to agriculture when the parcel of land supports a productive stand of timber. In contrast to McConnell et al. (1983), the authors assume that the landowner starts with a stand of trees of a particular age (so they relax the bare land assumption) and the goal of the landowner is to determine the age of timber harvest (and therefore the timing of land use conversion) under this scenario. It is also assumed that stumpage prices, regeneration costs and agriculture rent remain constant over time. Armstrong and Philips (1989) show that failure to separate forest bare land values from the productive value of standing timber can bias decisions towards immediate conversion. Yet, the authors do not examine the implications for the traditional Faustmann strategy of converting forestland into an alternative use at some point in the future or how evolving prices may affect forest management practices.

harvesting decisions are affected when we consider evolving prices, allowing for varying optimal rotation lengths over time.

Klemperer and Farkas (2001) also examine the impacts on optimal timber rotations when future land use changes. The study assumes nevertheless that landowners project a postharvest change in land use with a land value equal to or exceeding today's Faustmann value and that this future land value is independent of their assumed values for annual costs, annual revenues, stumpage prices, and income taxes. Under this context, it is shown that incorporation of high market valuations of timberland in the optimal harvest condition derived from the Faustmann formulation results in significantly reduced optimal rotation ages. Moreover, the impact of changes in the preceding variables on optimal rotations may be substantially greater, and sometimes in the opposite direction, compared to the Faustmann case.

In contrast to Klemperer and Farkas (2001), we examine the case where the original landowner can switch to a more profitable postharvest land use without selling the land. Moreover, we assume that the value of the alternative land use is rising (rather than fixed) over time. In contrast to McConnell et al. (1983) and Burgess and Ulph (2001) our alternative land use to forestry is urban and timber prices and regeneration costs are constant over time. This in turn allows us to discuss how rising urban rents affect forest management practices and explain the conversion of private working forests in an urbanizing setup.

### 3. Model

#### 3.1. Assumptions

Consider a single landowner holding a plot of bare land of fixed size  $\bar{L}$ . Assume that the entire plot is under commercial forestry at  $t=0$  and that it will be converted to residential use after two rotation cycles.<sup>3</sup> Let  $t$  represent the calendar time. Denote switching costs,  $S$ , as the cost per unit of land of switching from timber production to residential use. Development is assumed to be irreversible so that conversion of land from residential use back to forestry is economically infeasible.

There are two types of timber cuts. A regeneration cut consists on harvesting the current stand and providing for regeneration of the subsequent stand. A conversion cut harvests the current stand with no provision for regenerating a future stand and land is converted into residential use. We denote as  $T$  the rotation length of the timber stand or the age of the trees at the first harvest (regeneration cut) and  $D - T$  the rotation length of the timber stand at the second harvest (conversion cut), where  $D$  stands for the conversion cut date.

Stumpage prices,  $p$ , and planting costs,  $c$ , are constant over time and  $v(t)$  represents a strictly concave production function of wood per unit of land as a function of the age of the current stand. Initial planting costs equal  $c_0$ .

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<sup>3</sup> An endogenous number of rotation cycles could be imposed without changing any of the paper's results. Since our interest is in the sequence of harvest dates given a number of timber harvests, rather than on the number of timber harvests itself, we assume two rotation cycles to make the problem more tractable and easier for interpretation. In addition, given discounting and the long production period in forestry, the impacts of increases in the alternative use rent are likely more important on initial rotation lengths.



Residential rental rates are expected to increase at a constant rate such that  $R(t) = R(0)e^{\mu t}$ , where  $i > \mu > 0$  is the expected rate of change of residential rental rates and  $i$  is the discount rate.

### 3.2. The Landowner's Problem

When forestland is converted to residential use immediately after the second harvest, the landowner receives the discounted forest rents until the time of conversion plus the discounted residential rents thereafter, net the discounted conversion costs. Thus, a landowner chooses a forest management plan  $\{T, D\}$  that maximizes

$$V(T, D) = \bar{L} \left\{ -c_0 + [pv(T) - c]e^{-iT} + pv(D-T)e^{-iD} \right\} + \bar{L} \left\{ \int_D^{+\infty} R(0)e^{(\mu-i)t} dt - Se^{-iD} \right\} \quad (1)$$

Let  $\{T^*, D^*\}$  represent the landowner's optimal management plan. The necessary conditions for an interior local maximum at  $\{T^*, D^*\}$ , after some simplifications, are:

$$V_T = 0 \Leftrightarrow pv_t(T)e^{-iT} + ice^{-iT} = ipv(T)e^{-iT} + pv_t(D-T)e^{-iD} \quad (2)$$

$$V_D = 0 \Leftrightarrow pv_t(D-T)e^{-iD} - ipv(D-T)e^{-iD} = (R(0)e^{\mu D} - iS)e^{-iD} \quad (3)$$

at  $\{T^*, D^*\}$ .

Equation (2) defines the optimal condition for the first timber harvest. The left-hand side of equation (2) is the marginal benefit from waiting one more year. The marginal benefit consists of the extra amount of money earned because of the larger stand volume from the first timber crop plus the gain from postponing the payment of planting costs. On the right-hand side is the marginal cost of waiting one more year. The cost of waiting

comprises the forgone interest returns from harvesting immediately plus the foregone discounted marginal revenue product from the next harvest.

Condition (3) is the optimal conversion cut condition. A parcel should be converted to residential use when the net benefit of postponing conversion one year equals the net cost from postponing conversion. The net benefit includes the present value of the gain in stumpage value from added timber growth net the interests forgone from delaying second harvest timber revenues one year. The net cost represents the discounted value of residential land rent at time  $D$  net the switching costs savings that accrue from postponing the switch to residential use one year.

The second-order conditions can be expressed as

$$V_{TT} = e^{-iT} \left\{ pv_{tt}(T) - 2ipv_t(T) + i^2[pv(T) - c] \right\} + \bar{L}pv_{tt}(D - T)e^{-iD} \quad (4)$$

$$V_{DD} = \left\{ pv_{tt}(D - T) - 2ipv_t(D - T) + i^2pv(D - T) - (\mu - i)R(0)e^{\mu D} - i^2S \right\} e^{-iD} \quad (5)$$

$$|H| = V_{TT}V_{DD} - V_{DT}V_{TD} \quad (6)$$

where  $|H|$  represents the determinant of the Hessian matrix and by Young's theorem

$V_{TD} = V_{DT} = p[iv_t(D - T) - v_{tt}(D - T)]e^{-iD}$ . By strict concavity of the stand growth curve  $v(t)$ ,  $V_{TT} < 0$ ,  $V_{DD} < 0$ , and  $V_{TD} = V_{DT} > 0$ . Thus, regeneration and conversion cuts are complements in the land profit function  $V(T, D)$ . In order to assure that that  $\{T^*, D^*\}$  is a global maximum, we assume that  $|H| > 0$  holds.

Solving (2) and (3) for  $\{T^*, D^*\}$  yields the optimal regeneration cut date and the optimal conversion cut date, respectively, as

$$T^*(R(0), \mu, S, c_0, c, p, i) \quad (7)$$

$$D^*(R(0), \mu, S, c_0, c, p, i) \quad (8)$$

*The choice of converting after two rotation periods*

The optimality of our results so far is contingent on the number of rotations. However, if it's optimal to develop the land after two harvesting cuts then the following condition must be met:

$$\bar{L} \int_0^{+\infty} R(0)e^{(\mu-i)t} dt < V^F(T^F) < V(T^*, D^*) \quad (9)$$

where  $V^F(T^F)$  is the optimal value of forestland under the Faustmann case and  $T^F$  is the optimal Faustmann harvesting cut date. Increasing rents over time ensures that the gross returns to development are also rising over time. This in turn implies that forest land will eventually be developed at some future time, even if at time 0 (today) we have

$$\frac{R(0)}{i - \mu} < \frac{pv(T^F) - c}{e^{iT^F} - 1} - c_0.$$

*Harvesting Cut Dates in the Faustmann Case*

When there is no switch from commercial forestry to residential use while assuming identical, perpetually repeating rotations under even-aged management, the landowner chooses regeneration cuts in order to maximize the present value of land from continued management of the stand for timber production

$$V^F(T) = \bar{L} \left\{ -c_0 + \frac{pv(T) - c}{e^{iT} - 1} \right\} \quad (10)$$

The solution to this problem is the well know Faustmann regeneration cut date,  $T^F$ , where  $T^F$  satisfies the following equilibrium condition

$$\begin{aligned} pv_t(T) &= i(pv(T) - c) + i \frac{pv(T) - c}{e^{iT} - 1} \\ &= i(pv(T) - c) + \left[ pv(T) + \frac{pv(T) - ce^{iT}}{e^{iT} - 1} \right] \frac{i}{e^{iT}} \end{aligned} \quad (11)$$

According to (11) it is optimal to cut a stand, when the timber gains from delaying harvest compensate for the financial opportunity cost of leaving the trees standing plus the value lost from delaying all future rotations, captured by the rental value of the site.

#### 4. Rotation Lengths: Forestry in Perpetuity versus Developable Forestland

The optimal rotation lengths in our model satisfy the following condition:<sup>4</sup>

$$D^* - T^* \begin{matrix} \geq \\ < \end{matrix} T^* \text{ if } R(0)e^{uD^*} - iS \begin{matrix} \leq \\ > \end{matrix} pv_t(D^* - T^*)e^{-i(D^* - T^*)} - ic \quad (12)$$

According to (12) the optimal harvest lengths increase (decrease) over time if the opportunity cost of delaying the first harvest ( $pv_t(D^* - T^*)e^{-i(D^* - T^*)} - ic$ ) is higher (lower) than the opportunity cost of delaying the second harvest ( $R(0)e^{uD^*} - iS$ ). If both opportunity costs are the same, then rotation lengths do not change over time. Yet, this is not the Faustmann solution since the plot does not stay in commercial forestry in perpetuity. In fact, the appendix establishes that

$$D^* - T^* < T^F, \quad T^* < T^F \Rightarrow D^* < 2T^F \quad (13)$$

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<sup>4</sup> Using the first order conditions (2) and (3) and by concavity of  $v(t)$  we obtain result (12).

indicating that when there is the possibility to switch from forestry to a more valued alternative use in the future, a forest landowner will not adopt the Faustmann solution. As shown in (12) the optimal rotation lengths can be increasing, decreasing or constant over time. However, (13) shows that, no matter the case, both rotations will always be shorter than the Faustmann rotation lengths. The intuitive explanation for the above results is that whenever urban rents are increasing over time, the landowner finds it more profitable to have shorter harvest lengths because the relative opportunity cost of later harvests will have increased.

## 5. Comparative static analysis

With the equilibrium condition established, the stage is now set for comparative static analysis. The Faustmann case is briefly discussed first, with attention then turning to the case where forestland is converted to residential use at some point in the future.

The goal of our comparative static analysis is to examine how harvesting cuts ( $T^*$  and  $D^*$ ) depend on the exogenous parameters related to both the current and the alternative use of land and contrast the results with the effects in the Faustmann setup. All mathematical proofs are presented in the Appendix.

### *Faustmann*

The impacts of a higher stumpage price, higher interest rate and higher planting costs on  $T^F$  are respectively

$$\frac{dT^F}{dp} < 0, \quad \frac{dT^F}{di} < 0, \quad \frac{dT^F}{dc} > 0 \quad (14)$$

The Faustmann rotation is *ceteris paribus* shorter, the higher the timber price and interest rate and the lower the planting costs.<sup>5</sup>

#### *Changes in $R(0)$ , $\mu$ , and $S$*

In order to evaluate the impact of higher  $R(0)$ ,  $\mu$ , and  $S$  the sign of  $dT^*/d\theta$  and  $dD^*/d\theta$ , with vector  $\theta \equiv (R(0), \mu, S)$ , must be determined. As shown in the appendix

$$\frac{dT^*}{dR(0)} = \frac{V_{DR(0)}V_{TD}}{|H|} < 0, \quad \frac{dT^*}{d\mu} = \frac{V_{D\mu}V_{TD}}{|H|} < 0, \quad \frac{dT^*}{dS} = \frac{V_{DS}V_{TD}}{|H|} > 0 \quad (15)$$

$$\frac{dD^*}{dR(0)} = \frac{V_{DR(0)}V_{TT}}{|H|} < 0, \quad \frac{dD^*}{d\mu} = \frac{V_{D\mu}V_{TT}}{|H|} < 0, \quad \frac{dD^*}{dS} = \frac{V_{DS}V_{TT}}{|H|} > 0 \quad (16)$$

where the signs indicate the direction of the shift in timber cuts as parameters change.

According to (15) and (16), reductions in the optimal regeneration cut date and in the optimal conversion cut date associated with increasing residential rent, and decreasing conversion costs suggest that landowners reduce active forest management of their land as urbanization progresses. Note that regeneration cuts signal intentions to keep the land in forestry while conversion cuts signal the opposite. Higher residential bid rents increase the profitability of land in residential use, making it more costly to postpone timber harvesting cuts. Therefore, higher opportunity costs of delaying a future timber crop would lead to a younger harvest age for the first timber crop and anticipate conversion. A decrease in switching costs works qualitatively in a similar fashion by increasing the opportunity cost of delaying timber harvesting.

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<sup>5</sup> These results are well known and are not covered in detail here (see for example Hartwick and Olewiler (1998)).

#### *Changes in $c_0, c$*

A higher regeneration cost,  $c$ , postpones both harvesting cuts and therefore tenure in forestry is increased. Yet, an increase in the initial planting cost as a sunk cost,  $c_0$ , has no impact on harvesting cuts, while reducing the present value of the forest investment. These results are established in the appendix, where it is shown that

$$\frac{dT^*}{dc} = \frac{-V_{Tc}V_{DD}}{|H|} > 0, \quad \frac{dD^*}{dc} = \frac{V_{Tc}V_{DT}}{|H|} > 0 \quad (17)$$

$$\frac{dV(T^*, D^*)}{dc_0} = -1, \quad \frac{dT^*}{dc_0} = \frac{dD^*}{dc_0} = 0 \quad (18)$$

#### *Changes in $p$*

We now examine the impact of higher stumpage price on optimal forest management practices. In the classical Faustmann model a higher stumpage price would lead to an unambiguous decrease in all optimal regeneration cut dates, that is, to a decrease in  $T^F$  (see (14)). A higher stumpage price increases the profitability of harvesting, making it more costly to leave the stand standing one more year. However, with evolving residential rents and a land use change at rotation-end, the impact of a one-time change in stumpage price on optimal rotations cannot be generalized as in the Faustmann case.

Deriving the impact of a higher  $p$  on harvesting cut dates is more difficult in our framework than the analogous Faustmann calculations since the effect on the current stand value must be considered along with the opposite effect that operates through the land expectation value at the beginning of the second harvest.

As shown in the appendix,

$$\frac{dT^*}{dp} = \frac{V_{TD}(V_{Tp} + V_{Dp}) + V_{Tp}\mu R(0)e^{(\mu-i)D^*}}{|H|} > 0 \quad (19)$$

$$\frac{dD^*}{dp} = \frac{V_{TD}(V_{Tp} + V_{Dp}) + V_{Dp}(ipv_t(T^*) - pv_{tt}(T^*))e^{-iT^*}}{|H|} > 0 \quad (20)$$

when  $V_{Tp} + V_{Dp} > 0$  and  $T^* \leq D^* - T^*$ , where

$$V_{Tp} = \frac{-ice^{-iT^*}}{p} < 0 \text{ and } V_{Dp} = \frac{(R(0)e^{uD^*} - iS)e^{-iD^*}}{p} > 0.^6 \quad (21)$$

The possible outcomes for the sign of (19) and (20) depend on the relative importance of the stumpage price to the marginal values of the regeneration cut ( $V_{Tp}$ ) and conversion cut ( $V_{Dp}$ ) and on whether rotation lengths are increasing or decreasing over time.

Therefore,

(i) if  $V_{Tp} + V_{Dp} < 0$ , the forgone interest returns from postponing the regeneration cut ( $ipv(T^*)e^{-iT^*} + ipv(D^* - T^*)e^{-iD^*}$ ) are higher than the extra timber revenue earned from a larger stand volume from postponing the regeneration cut ( $pv_t(T^*)e^{-iT^*}$ ). On the other hand, the opportunity cost of holding forestland is negative ( $(R(0)e^{uD^*} - iS)e^{-iD^*} - ice^{-iT^*} < 0$ ). Thus, a higher stumpage price would shorten the optimal regeneration cut ( $\frac{dT^*}{dp} < 0$ ) while having an ambiguous effect on the optimal conversion date ( $\frac{dD^*}{dp} > 0$ ). However, if it is the case where  $T^* \leq D^* - T^*$ , then  $\frac{dD^*}{dp} < 0$

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<sup>6</sup> Note that  $V_{TD} > 0$  and  $|H| > 0$  owing to the second-order sufficiency conditions for the optimal forest management.



and  $\frac{dT^*}{dp} < 0$  (see the appendix for details). This scenario is likely to occur when urban rents today or/and the rate of increase in urban rents are very low such that the impact of a change in  $p$  on the cost of holding the land is outweighed by the impact of a change in stumpage price on the cost of holding the trees.

(ii) if  $V_{Tp} + V_{Dp} > 0$  implying that  $(R(0)e^{uD^*} - iS)e^{-iD^*} - ice^{-iT^*} > 0$ , then a higher stumpage price has an ambiguous effect on the optimal regeneration cut date ( $\frac{dT^*}{dp} > 0$ ) while postponing the optimal conversion cut date ( $\frac{dD^*}{dp} > 0$ ). In this case the gain from switching later into residential use outweighs the gain from postponing the payment of planting costs while the marginal revenue benefit from postponing the regeneration cut outweighs the foregone interests returns from delaying both timber cut dates.

However, if  $\mu = 0$  but  $R(0) - iS > 0$ , it is also optimal to delay the first harvest date ( $\frac{dT^*}{dp} > 0$ ) and thus depart from the Faustmann result.

Moreover, if  $T^* \geq D^* - T^*$  is satisfied, we also show in the appendix that we have both  $\frac{dT^*}{dp} > 0$  and  $\frac{dD^*}{dp} > 0$  when  $V_{Tp} + V_{Dp} > 0$  and  $\mu > 0$ .

The prospect of switching to a more valuable land use at the end of the second rotation provides an incentive to delay the first harvest and time the second harvest according to this information despite the increase in stumpage price at time 0. The scenario where  $\frac{dT^*}{dp} > 0$  and  $\frac{dD^*}{dp} > 0$  is thus likely to occur when urban rents today or/and the rate of increase in urban rents are very high such that the impact of a change in

$p$  on the cost of holding the land outweighs the impact of a change in stumpage price on the cost of holding the trees.

Finally,

(iii) if  $V_{Tp} + V_{Dp} = 0$ , then higher stumpage price would unambiguously shorten the optimal regeneration cut date ( $\frac{dT^*}{dp} < 0$ ) and delay the optimal conversion cut date ( $\frac{dD^*}{dp} > 0$ ). In this case, if the stumpage price increases, then the relative profitability of timber and residential use changes in favor of timber production. This result is independent of the behavior of rotation lengths over time.

*Change in  $i$*

Finally, to examine the impact of higher discount rate, the sign of  $\frac{dT^*}{di}$  and  $\frac{dD^*}{di}$  must be evaluated. Like in the previous case, the impact of a higher interest rate on harvesting cuts cannot be definitely determined. Only when  $V_{Ti}$  and  $V_{Di}$  have the same sign it is possible to have unambiguous effects. As shown in the appendix,

$$\frac{dT^*}{di} = \frac{-V_{Ti}V_{DD} + V_{Di}V_{TD}}{|H|} \begin{matrix} > \\ < \end{matrix} 0 \quad \text{when } V_{Di} \begin{matrix} > \\ < \end{matrix} 0 \quad \text{and } V_{Ti} \begin{matrix} > \\ < \end{matrix} 0 \quad (22)$$

$$\frac{dD^*}{di} = \frac{-V_{Di}V_{TT} + V_{Ti}V_{TD}}{|H|} \begin{matrix} > \\ < \end{matrix} 0 \quad \text{when } V_{Di} \begin{matrix} > \\ < \end{matrix} 0 \quad \text{and } V_{Ti} \begin{matrix} > \\ < \end{matrix} 0 \quad (23)$$

where

$$V_{Ti} = (D^* - T^*)pv_t(D^* - T^*)e^{-iD^*} - (pv(T^*) - c)e^{-iT^*} \quad (24)$$

$$V_{Di} = \left( \frac{R(0)e^{uD^*} - pv_t(D^* - T^*)}{i} \right) e^{-iD^*} \quad (25)$$

Next, we present the conditions under which  $V_{Ti}$  and  $V_{Di}$  have the same sign.

We show in the appendix that a necessary condition for  $V_{Ti} > 0$  is that  $D^* - T^*$  is “significantly” smaller than  $T^F$ .<sup>7</sup> Moreover, a sufficient condition for  $V_{Di} > 0$  is that optimal harvest lengths do not increase over time ( $D^* - T^* \leq T^*$ ). This implies, according to (12), that the opportunity cost of delaying the second harvest is at least as large as the opportunity cost of delaying the first harvest. Therefore, if  $D^* - T^* \leq T^* < T^F$  is satisfied, it is optimal to delay both regeneration and conversion cut dates, following an increase in the discount rate. In contrast to the Faustmann case, the private landowner may optimally choose to postpone receiving timber revenues following the increase in the relative preference for the present when the opportunity cost of delaying the first harvest is larger than the opportunity cost of delaying the second harvest cut date.

In contrast, a necessary condition for  $V_{Ti} < 0$  is that  $D^* - T^*$  is close to  $T^F$ . Hence, if  $V_{Ti} < 0$ , implying that optimal harvest lengths decrease over time ( $D^* - T^* \geq T^*$ ), it is also the case that  $V_{Di} < 0$ .<sup>8</sup> We then may conclude that if  $T^F > D^* - T^* \geq T^*$  occurs it is optimal to await both regeneration and conversion cut dates following an increase in the discount rate. This result is similar to the Faustmann comparative static result, where

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<sup>7</sup> We assume a given  $T^F$ , which depending on the trees species may be large or small. Therefore, if  $T^F$  is low, that is, if trees are fast growing, then the length of the second rotation, which by (13) is such that  $D^* - T^* < T^F$ , may be so small that it turns out to be meaningless. Thus, this case is more relevant for large  $T^F$ , that is, for slow growing species. In this case it is more likely to obtain results different from Faustmann’s.

<sup>8</sup> The proofs are presented in the appendix.

an increase for the relative preference for the present anticipates regeneration cuts and thus, determines shorter rotation lengths. However, and in contrast to the Faustmann case where the optimal rotation lengths are identical, when there is the possibility to switch to a more valuable alternative use in the future, despite that  $V_{Ti} < 0$  as in the Faustmann solution, rotation lengths are non-decreasing over time, that is,  $D^* - T^* \geq T^*$ .<sup>9</sup>

The intuition for our comparative statics results of a marginal change in the discount rate on the optimal harvesting cuts can also be given by discussing the impact of a marginal change in  $i$  on the opportunity cost of holding the stand value and on the opportunity cost of holding the land. Because these effects run in opposite directions and depend on market conditions, one cannot have simple unambiguous results as in the Faustmann case.

In the traditional Faustmann model, the opportunity cost of land is the discounted value of future rotations. If harvest in a rotation is delayed, the harvests in all future rotations will also be delayed, which reduces the present value of future rotations. By inspection of (11), an increase in the discount rate reduces the Faustmann optimal harvest age as the opportunity cost of delaying harvest increases. The land expectation value is also constant from timber crop to timber crop since it is assumed that stumpage price, stand volume, regeneration cost and the interest rate would repeat themselves from harvest to harvest. Therefore, if the discount rate rises, the Faustmann optimal rotation

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<sup>9</sup> Recall that  $V_{Ti}^F(T^F) < 0$ .

length declines since land devoted to forestry becomes less valuable (see the comparative statics (14)).<sup>10</sup>

When market conditions or government policies that affect conversion costs change over time, the land expectation value is no longer constant from timber crop to timber crop. As a result, the optimal harvest age for any particular crop depends of its own stand value and the land expectation value immediately after the harvest (see equilibrium conditions (2) and (3)).

Combining the equilibrium conditions (2) and (3) yields, after some simplifications, the following expression

$$pv_t(T) = i(pv(T) - c) + \left[ pv(D - T) + \frac{R(0)e^{uD}}{i} - S \right] \frac{i}{e^{i(D-T)}} \quad (26)$$

According to (26), the optimal harvest age for the first timber crop depends on the current stand value and its increment plus the land expectation value at the beginning of the second crop. The result in equation (26) together with (12) also suggests that the land expectation value may differ from the Faustmann case (see the equilibrium condition (11)) when residential rents increase over time. If after the second rotation, residential development is more valuable than forestry,  $\frac{R(0)e^{uD}}{i} - S$  replaces  $\frac{pv(T) - ce^{iT}}{e^{iT} - 1}$  in the land expectation value at the beginning of the second crop, even if optimal rotation lengths are constant over time. Under this scenario, it does not necessarily follow that a

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<sup>10</sup> Note that  $\frac{pv(T) - c}{e^{iT} - 1}$  depends on the discount factor and will decrease exponentially with increases in  $i$ .

Moreover, from (11) the marginal cost of delaying the regeneration cut includes the cost of holding the trees and the cost of holding the land. Because the Faustmann model assumes that land has no value other than for growing trees, the value of holding the land in forestry is equal to the net timber revenues from infinite harvests with equal rotations.

high interest rate results in accelerated regeneration cuts and conversion of forestland. In fact, depending on the residential rent at time 0, its growth rate  $\mu$  and conversion costs it maybe the case that it payoffs for the landowner to postpone both the regeneration cut and the conversion cut dates, following an increase in the discount factor. This can occur if the impact of the increase in  $i$  on the cost of holding the land outweighs the impact of  $i$  on the cost of holding the trees.<sup>11</sup> A higher interest rate for the first timber crop would shorten the regeneration cut date while a higher interest rate for the second timber crop and residential value would lower the land expectation value at the beginning of the second rotation period and thus lengthen the harvest age for the first timber crop. Since these two effects work at cross purposes, the net effect of these two forces on the cost of delaying the first rotation date is actually an empirical question.

## 5. Conclusions

Despite a large and growing literature in forestry economics, remarkably little theoretical work has been done on forest management practices and deforestation at the urban periphery. This paper has sought to complete the discussion regarding the effects of a future land use change on forest management decisions. Its primary contribution is to examine how changes on urban growth factors, timber prices and input costs affect regeneration and conversion cut dates in the urban-rural interface.

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<sup>11</sup> The right-hand-side of (26) can be re-written as  $i(pv(T) - c) + \frac{i}{e^{iT}}(pv(T) - S) + \frac{R(0)e^{\mu T}}{e^{(i-\mu)T}}$ . While in the Faustmann case,  $i(pv(T) - c) > \frac{i}{e^{iT}}(pv(T) - c) + \frac{i}{e^{iT} - 1}(pv(T) - c)e^{-iT}$  regardless of the value of  $i$ , in our generalized Faustmann model a similar relationship based on (26) depends on the values of  $\mu$ ,  $R(0)$  and  $S$ .

We show that with evolving urban rents and a land use change at rotation-end, the impact of a one-time change in stumpage price or on the discount rate on optimal rotations cannot be generalized as in the Faustmann case. Moreover, reductions in the optimal regeneration cut date and in the optimal conversion cut date associated with increasing urban rents and decreasing conversion costs suggest that landowners reduce active forest management of their land as urbanization progresses.

Our results imply nevertheless that planting more valuable tree species in areas under urban growth pressure can postpone regeneration cuts and forestland conversion. Similarly, policies that increase conversion costs can also prevent deforestation in suburban and ex-urban areas where urban sprawl is a stressing problem.

There are possible extensions to our model which we did not consider. We assume that the landowner's decisions are taken in a deterministic environment. Future work could examine how regeneration and conversion cuts change when urban rents are driven by uncertainties in the housing market. Another possible extension would be to include forest management investment into the model. Silvicultural investment clearly need not remain fixed in face of shifting prices and prices expectations. The added complexity of the expanded model undoubtedly would complicate the comparative static analysis greatly that probably few meaningful results can be extracted and numerical simulations may be in order. In spite of these limitations, our model provides a useful step in explaining the trends in the use and management of private land for timber production at the urban-rural interface.

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## Appendix

### Deriving (13)

If it is optimal to convert the parcel to residential use at time  $t = D^*$ , it has to be the case that rent in residential use exceeds the forestry rent forgone plus the opportunity cost of the capital needed to convert the land net the savings from not incurring planting costs, that is

$$R(D^*, \cdot) > i \frac{pv(T^F) - c}{e^{iT^F} - 1} + iS - ic \quad (A1)$$

From (3) and (11), while using (A1) we have that  $pv_t(D^* - T^*) - ipv(D^* - T^*) > pv_t(T^F) - ipv(T^F)$  and from concavity of  $v(t)$  it follows that  $D^* - T^* < T^F$ . This in turn implies that  $pv_t(D^* - T^*)e^{-i(D^* - T^*)} > pv_t(T^F)e^{-iT^F}$ . From (2) we can easily obtain  $pv_t(T^*) - ipv(T^*) + ic > pv_t(T^F) - ipv(T^F) + ic$ . Finally, again from concavity of  $v(t)$ ,  $T^* < T^F$ . Since  $D^* - T^* < T^F$  and  $T^* < T^F$  it follows that  $D^* < T^F + T^* < 2T^F$ .

### Deriving (15), (16) and (17)

Totally differentiating (2) and (3) with respect to  $T$ ,  $D$  and  $\theta$  dividing the resulting system of equations by  $d\theta$ , and applying Cramer's rule yields

$$\frac{dT^*}{d\theta} = \frac{V_{D\theta}V_{TD}}{|H|} \quad (A2)$$

$$\frac{dD^*}{d\theta} = \frac{-V_{D\theta}V_{TT}}{|H|} \quad (A3)$$

Also, for  $\theta = R(0), \mu, S$  we get  $V_{DR(0)} = -e^{(\mu-i)D^*} < 0, V_{TR(0)} = 0, V_{D\mu} = -D^*R(0)e^{(\mu-i)D^*} < 0, V_{T\mu} = 0, V_{TS} = 0$ , and  $V_{DS} = ie^{-iD^*} > 0$ . Given that  $V_{TD} > 0$ , and  $|H| > 0$ , for  $V_{D\theta} < 0$ ,  $\frac{dT^*}{d\theta} < 0$ , while for  $V_{D\theta} > 0$ ,  $\frac{dT^*}{d\theta} > 0$ . Since  $V_{TT} < 0$ , for  $V_{D\theta} < 0(> 0)$ ,  $\frac{dD^*}{d\theta} < 0(> 0)$ . Since  $V_{Tc} = ie^{-iT^*} > 0$ , and  $V_{Dc} = 0$ , for  $\theta = c$ , we obtain our result (17).

### Deriving (19) and (20)

By differentiating totally (3) with respect to  $\theta = p$ , we obtain

$$V_{Dp} = -ie^{-iD}v(D^* - T^*) + v_t(D^* - T^*)e^{-iD^*} \quad (A4)$$

$$\text{Using } V_D = 0 \text{ in (A4), yields } V_{Dp} = \frac{R(0)e^{(\mu-i)D^*} - iSe^{-iD^*}}{p}. \quad (A5)$$

By differentiating (2) with respect to  $\theta = p$ , we get

$$V_{Tp} = v_t(T^*)e^{-iT^*} - ie^{-iT^*}v(T^*) - v_t(D^* - T^*)e^{-iD^*} \quad (A6)$$

Using  $V_T = 0$  in (A6) we get, after some simplifications

$$V_{Tp} = -\frac{ice^{-iT^*}}{p} < 0. \quad (A7)$$

Using  $V_D = 0$  in  $V_{DD}$ , and simplifying we obtain

$$V_{DD} = -V_{TD} - \mu R(0)e^{(\mu-i)D^*} < 0 \quad (A8)$$

while using  $V_T = 0$  in  $V_{TT}$ , and simplifying we have

$$V_{TT} = -V_{DT} - ipv_t(T^*)e^{-iT^*} + pv_{tt}(T^*)e^{-iT^*} < 0. \quad (A9)$$

Applying Cramer's rule, we obtain

$$\frac{dT^*}{dp} = \frac{-V_{Tp}V_{DD} + V_{Dp}V_{TD}}{|H|} \quad (A10)$$

$$\frac{dD^*}{dp} = \frac{-V_{Dp}V_{TT} + V_{Tp}V_{TD}}{|H|}. \quad (A11)$$

Therefore, substituting (A5), (A7) and (A8) into (A10) and collecting terms yields (19). Similarly, substituting (A5), (A7) and (A9) into (A11) and collecting terms yields (20).

### Sign of (A5)

To determine the sign of (A4), let  $D - T = T^{Fs}$ , where  $T^{Fs}$  stands for the Fisher solution, that is, the optimal harvest time in the single crop case.<sup>12</sup> In this case, we obtain that (A4) evaluated at  $D - T = T^{Fs}$  is zero, that is,  $V_{Dp}(D - T = T^{Fs}) = 0$ . By concavity of  $v(t)$ , we have that  $T^F < T^{Fs}$ , implying that  $V_{Dp}(T^F) > 0$ . Since  $D^* - T^* < T^F$  by (13), it implies that  $V_{Dp}(D^* - T^*) > 0$ , as  $\frac{pv_t(D^* - T^*)}{pv(D^* - T^*)} > i$ . Thus, we conclude that the sign of (A5) must be positive,  $V_{Dp} = \frac{\{R(0)e^{\mu D^*} - iS\}e^{-iD^*}}{p} > 0$ .

### Deriving the signs of (19) and (20)

$$\text{Let } V_{Dp} + V_{Tp} = \frac{\{R(0)e^{\mu D^*} - iS\}e^{-iD^*}}{p} - \frac{ice^{-iT^*}}{p} > 0 \quad (A12)$$

By adding  $\frac{ice^{-iT^*}}{p}$  and adding and subtracting  $pv_t(D^* - T^*)e^{-iD^*}$  in (A12), we obtain

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<sup>12</sup> Hence, trees should be cut when the timber value of standing trees grows at the interest rate, that is, as fast as the return on the alternative asset. It is as if the land has no opportunity cost, as in the case of "frontier land".

$$\begin{aligned} & \frac{\{R(0)e^{\mu D^*} - iS\}e^{-iD^*}}{p} - \frac{ice^{-iT^*}}{p} + \frac{ice^{-iT^*}}{p} - \\ & - \frac{pv_t(D^* - T^*)e^{-iD^*}}{p} + \frac{pv_t(D^* - T^*)e^{-iD^*}}{p} > 0 \end{aligned} \quad (A13)$$

Therefore,

$$\frac{\{R(0)e^{\mu D^*} - iS\}e^{-iD^*}}{p} + \frac{ice^{-iT^*}}{p} - \frac{pv_t(D^* - T^*)e^{-iD^*}}{p} > 0 \quad (A14)$$

and

$$\frac{pv_t(D^* - T^*)e^{-iD^*}}{p} - \frac{ice^{-iT^*}}{p} > 0 \quad (A15)$$

are sufficient conditions for (A12) to be positive. Note that (A15) has to be always positive, since from (2) only in this case  $T^*$  that solves (2) is lower than  $T^F$ . Moreover, from (12), (A14) implies that  $T^* \geq D^* - T^*$ .

$$\text{If } V_{Dp} + V_{Tp} = \frac{\{R(0)e^{\mu D^*} - iS\}e^{-iD^*}}{p} - \frac{ice^{-iT^*}}{p} < 0 \quad (A16)$$

by adding and subtracting  $\frac{ice^{-iT^*}}{p}$  and subtracting  $pv_t(D^* - T^*)e^{-iD^*}$  in (A16), yields

$$\frac{\{R(0)e^{\mu D^*} - iS\}e^{-iD^*}}{p} + \frac{ice^{-iT^*}}{p} - \frac{pv_t(D^* - T^*)e^{-iD^*}}{p} - 2\frac{ice^{-iT^*}}{p} < 0 \quad (A17)$$

Therefore,

$$\frac{\{R(0)e^{\mu D^*} - iS\}e^{-iD^*}}{p} + \frac{ice^{-iT^*}}{p} - \frac{pv_t(D^* - T^*)e^{-iD^*}}{p} < 0 \quad (A18)$$

is a sufficient condition for (A16) to hold. By (12), (A18) implies that  $T^* \leq D^* - T^*$ . Also, we may rewrite (26), as follows:

$$v_t(T) = iv(T) + \left[ v(D - T)e^{-iD} + \frac{e^{iT}i}{p} \left( \frac{R(0)e^{(u-i)D}}{i} - Se^{-iD} - ce^{-iT} \right) \right] \quad (A19)$$

Hence, if the price of timber increases, and if the term inside brackets is positive, that is, if  $V_{Dp} + V_{Tp} > 0$ , that term is positive but smaller. By strict concavity of  $v(t)$ ,  $T^*$  is thus

larger, implying that  $\frac{dT^*}{dp} > 0$ . On the other hand, if the term inside brackets is negative,

that is, if  $V_{Dp} + V_{Tp} < 0$ , that term is positive but larger. By strict concavity of  $v(t)$ ,  $T^*$  is in this case smaller implying that  $\frac{dT^*}{dp} < 0$ . In this scenario, by inspection of (3), if  $T^*$  is

smaller then  $D^*$  has also to be smaller, by strict concavity of  $v(t)$ , implying that  $\frac{dD^*}{dp} < 0$ .

In summary, if  $V_{Dp} + V_{Tp} < 0$ ,  $\frac{dT^*}{dp} < 0$ , and  $\frac{dD^*}{dp} < 0$ , where  $T^* \leq D^* - T^*$ .

If  $V_{Dp} + V_{Tp} > 0$ ,  $\frac{dT^*}{dp} > 0$ , and  $\frac{dD^*}{dp} > 0$ , where  $T^* \geq D^* - T^*$ .

### Deriving (24) and (25)

By differentiating (2) with respect to  $\theta = i$  we get

$$V_{Ti} = -e^{-iT^*} (pv(T^*) - c) + iT^* e^{-iT^*} (pv(T^*) - c) + \\ + pv_t(D^* - T^*) e^{-iD^*} (D^* - T^*) e^{-iT^*} pv_t(T^*) \quad (A20)$$

Simplifying (A20) using  $V_T = 0$  yields (24) as

$$V_{Ti} = -e^{-iT^*} (pv(T^*) - c) + pv_t(D^* - T^*) e^{-iD^*} (D^* - T^*). \quad (A21)$$

Similarly, differentiating (3) with respect to  $\theta = i$  yields

$$V_{Di} = -D^* e^{-iD^*} pv_t(D^* - T^*) + (D^* - 1) e^{-iD^*} pv(D^* - T^*) + \\ + D^* e^{(\mu-i)D^*} R(0) + (1-i) e^{-iD^*} iS \quad (A22)$$

which simplifies to (25) as

$$V_{Di} = \frac{R(0) e^{(\mu-i)D^*} - pv_t(D^* - T^*) e^{-iD^*}}{i} = -e^{-iD^*} (pv(D^* - T^*) - S). \quad (A23)$$

### Deriving the sign of (24)

To determine the sign of (24) we compare it with the corresponding expression in the

Faustmann case, that is,  $V_{Ti}^F = pv(T^F) - c - \frac{e^{-iT^F} T^F i (pv(T^F) - c)}{1 - e^{-iT^F}} < 0$ . Using (11)

yields  $V_{Ti}^F(T^F) = -(pv(T^F) - c) + T^F e^{-iT^F} pv_t(T^F) < 0$ . Since

$$\frac{\partial(-(pv(T) - c) + T e^{-iT} pv_t(T))}{\partial T} = pv_{tt}(T) e^{-iT} T - i e^{-iT} pv_t(T) - pv_t(T) (1 - e^{-iT}) < 0,$$

by strict concavity of  $v(t)$ , we conclude that  $V_{Ti}^F(T)$  increases when  $T$  decreases from  $T^F$ . As  $D^* - T^* < T^F$ ,  $V_{Ti}^F(D^* - T^*)$  is either a larger negative number (or smaller in absolute value), or a positive number. In the first case, (24) evaluated at  $D^* - T^*$  is negative, that is

$$V_{Ti}(T^* = D^* - T^*) = \\ e^{-iT^*} \left\{ (pv(D^* - T^*) - c) + pv_t(D^* - T^*) e^{-i(D^* - T^*)} (D^* - T^*) \right\} < 0 \quad (A24)$$

while it is positive in the second case

$$V_{Ti}(T^* = D^* - T^*) = \\ e^{-iT^*} \left\{ (pv(D^* - T^*) - c) + pv_t(D^* - T^*) e^{-i(D^* - T^*)} (D^* - T^*) \right\} > 0 \quad (A25)$$

By inspection of (A24), and given strict concavity of  $v(t)$ , we may conclude that the closer is  $D^* - T^*$  to  $T^F$ , the more likely it is that (A24) holds, as the first term

dominates. Recall that  $V_{Ti}^F(T^F) < 0$ . In contrast, the far apart is  $D^* - T^*$  from  $T^F$ , that is, the smaller is  $D^* - T^*$ , the more likely it is that (A25) holds as it increases the likelihood that the second term dominates. Note that if  $D^* - T^*$  is small  $v_t(D^* - T^*)$  is large while  $v(D^* - T^*)$  is small by concavity of  $v(t)$ . Finally, given that  $v(t)$  is increasing, if (A24) holds, we conclude that  $D^* - T^* \geq T^*$  is a necessary and sufficient condition for  $V_{Ti} < 0$ , that is, for (24) to be negative (in order to guarantee that (24) is a larger negative number). In contrast, if (A25) holds, a necessary condition for  $V_{Ti} > 0$ , that is, for (24) to be positive is that  $(D^* - T^*)$  is “significantly” smaller than  $T^F$ . In this case, decreasing  $T^*$  from  $(D^* - T^*)$ , that is,  $T^* \leq D^* - T^*$ , keeps  $V_{Ti} > 0$ . In contrast, increasing  $T^*$  from  $(D^* - T^*)$ , that is,  $T^* > D^* - T^*$ , may not necessarily keep the sign of  $V_{Ti} > 0$ .

### Deriving the sign of (25)

By adding and subtracting  $Se^{-iD^*}$  and  $ce^{-iT^*}$  to (A23) we obtain

$$\frac{R(0)e^{(\mu-i)D^*} - pv_t(D^* - T^*)e^{-iD^*} + ice^{-iT^*} - iSe^{-iD^*}}{i} + Se^{-iD^*} - ce^{-iT^*}. \quad (\text{A26})$$

If the numerator in the first term in (A26) is positive (negative) and  $Se^{-iD^*} - ce^{-iT^*} > 0 (< 0)$ , it follows that  $V_{Di} > 0 (< 0)$ .

Hence

$$R(0)e^{(\mu-i)D^*} - pv_t(D^* - T^*)e^{-iD^*} + ice^{-iT^*} - iSe^{-iD^*} > 0 (< 0) \quad (\text{A27})$$

and

$$Se^{-iD^*} - ce^{-iT^*} > 0 (< 0) \quad (\text{A28})$$

are sufficient conditions for  $V_{Di} > 0 (< 0)$ , where the first condition implies from (12) that  $T^* \geq D^* - T^*$  ( $T^* \leq D^* - T^*$ ).

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